International Journal of Interdisciplinary Research in Arts and Humanities (IJIRAH)
Impact Factor: 5.225, ISSN (Online): 2456 - 3145
(www.dvpublication.com) Volume 8, Issue 2, 2023

SIMULTANEOUS CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Cite This Article: Yogeeta Narwal, "Simultaneous Conformable Fractional Differential Equations", International Journal of Interdisciplinary Research in Arts and Humanities, Volume 8, Issue 2, July - December, Page Number 7-13, 2023.

Abstract:

In this paper we give general solution of simultaneous conformable fractional differential equations. Since there are many definitions for fractional derivatives, we use the conformable derivative to get exact solutions. Here we are giving two methods for solving simultaneous fractional differential equations with constant coefficients. Some specific examples based on both methods are given as an application.

Key Words: Conformable, Simultaneous Equations, Conformable Simultaneous Differential Equations.

1. Introduction:

Many authors have solved many well-known differential equations like the Conformable Fractional Heat equation, Bessel equation, Legendre equation and many more. [1], [4], [5], [6], [7], [9] and [10]. However, there are many definitions available in the literature for fractional derivatives. The main ones are the Riemann-Liouville definition and the Caputo definition, see [8].

(i) Riemann-Liouville Definition. For $\alpha \in [n-1, n)$,the α derivative of *f* is

$$D_a^{\alpha}(f)(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

(ii) Caputo Definition. For $\alpha \in [n-1, n)$, the α derivative of *f* is

$$D_a^{\alpha}(f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}}dx$$

Such definitions have many setbacks such as

(i) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1)=0$ ($D_a^{\alpha}(1)=0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two

$$D_a^{\alpha}(fg)=fD_a^{\alpha}(g)+gD_a^{\alpha}(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two

$$D_a^{\alpha}(f/g)=\frac{gD_a^{\alpha}(f)-fD_a^{\alpha}(g)}{g^2}$$

(iv) All fractional derivatives do not satisfy chain rule

$$D_a^{\alpha}(fog)(t)=f^{(\alpha)}(g(t))g^{(\alpha)}(t)$$

(v) All fractional derivatives do not satisfy: $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$, in general.

(vi) All fractional derivatives, especially Caputo definition, assume that the function *f* is differentiable.

We refer the reader to [3] for more results on Caputo and Riemann - Liouville Definitions.

Recently, the authors in [2], gave a new definition of fractional derivative which is a natural extension to the usual first derivative. So many papers since then were written, and many equations were solved using this definition. The definition goes as follows:

Given a function $f:[0,\infty)\to\mathbb{R}$. Then, for all $t>0, \alpha\in(0,1)$, let

$$T_{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},$$

T_{α} is called the conformable fractional derivative of *f* of order α .

Let $f^{\alpha}(t)$ stands for $T_{\alpha}(f)(t)$.

If *f* is α -differentiable in some (0, b), b>0 and $\lim_{t\to 0^+} f^{\alpha}(t)$ exists, then define

$$f^{\alpha}(0)=\lim_{t\to 0^+} f^{\alpha}(t)$$

According to this definition, we have the following properties, [2]

1. $T_\alpha(1)=0$,
2. $T_\alpha(t^p)=pt^{p-\alpha}\ \forall p\in\mathbb{R}$
3. $T_\alpha(\sin at)=at^{1-\alpha}\cos at, a\in\mathbb{R}$
4. $T_\alpha(\cos at)=-at^{1-\alpha}\sin at, a\in\mathbb{R}$
5. $T_\alpha(e^{at})=at^{1-\alpha}\,\mathrm{e}^{at}, a\in\mathbb{R}$

Further, many functions behave as in the usual derivative. Here are some formulas

$$T_\alpha\left(\frac{1}{\alpha}t^\alpha\right)=1$$

$$T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right)=e^{\frac{1}{\alpha}t^\alpha},$$

$$T_\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right)=\cos\frac{1}{\alpha}t^\alpha$$

$$T_\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right)=-\sin\frac{1}{\alpha}t^\alpha$$

Preliminaries:

(i) $$T_\alpha(\log t)=\lim_{\varepsilon\to0}\frac{\log\left(t+\varepsilon t^{1-\alpha}\right)-\log(t)}{\varepsilon}$$

$$=\lim_{\varepsilon\to0}\frac{\log\left(1+\varepsilon t^{1-\alpha}\right)}{\varepsilon}$$

$$\left\{\because \log m-\log n=\log\left(\frac{m}{n}\right)\right\}$$

$$=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left[\varepsilon t^{-\alpha}-\frac{\left(\varepsilon t^{-\alpha}\right)^2}{2}+\frac{\left(\varepsilon t^{-\alpha}\right)^3}{3}-\dots\right] \qquad \left\{\because \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right\}$$

$$=t^{-\alpha}$$

$$\Rightarrow T_\alpha(\log t)=\frac{1}{t^\alpha}$$

Similarly, $I_\alpha\left(\frac{1}{t^\alpha}\right)=\int\frac{f(t)}{t^{1-\alpha}}dt=\int\frac{1}{t^\alpha t^{1-\alpha}}dt=\int\frac{1}{t}dt=\log t$

Thus, $I_\alpha\left(\frac{1}{t^\alpha}\right)=\log t$

(ii) $$T_\alpha\left(a^t\right)=\lim_{\varepsilon\to0}\frac{a^{t+\varepsilon t^{1-\alpha}}-a^t}{\varepsilon}$$

$$=\lim_{\varepsilon\to0}\frac{a^t\left[a^{\varepsilon t^{1-\alpha}}-1\right]}{\varepsilon}$$

$$\left\{\because a^t=1+t\log a+\frac{t^2}{2!}(\log a)^2+\dots\right\}$$

$$=\lim_{\varepsilon\to0}a^t\frac{1}{\varepsilon}\left[1+\varepsilon t^{1-\alpha}\,\mathrm{loga}+\frac{\left(\varepsilon t^{1-\alpha}\right)^2}{2!}(\mathrm{loga})^2+\dots-1\right]$$

$$= t^{1-\alpha} a^t \log a$$

Thus, $T_\alpha\left(a^t\right) = t^{1-\alpha} a^t \log a$

Similarly, $I_\alpha\left(t^{1-\alpha} a^t \log a\right) = \int \frac{t^{1-\alpha} a^t \log a}{t^{1-\alpha}} dt$

$$= \int a^t (\log a)\, dt$$

$$= \frac{a^t \log a}{\log a} = a^t$$

Thus, $I_\alpha\left(\mathrm{t}^{1-\alpha} \mathrm{a}^t \log a\right) = a^t$

(iii) $T_\alpha(\sinh t) = \lim_{\varepsilon \to 0} \frac{\sinh\left(t + \varepsilon t^{1-\alpha}\right) - \sinh t}{\varepsilon}$

$$= \lim_{\varepsilon \to 0} \frac{\left(e^{t+\varepsilon t^{1-\alpha}} - e^{-t-\varepsilon t^{1-\alpha}}\right) - \left(e^t - e^{-t}\right)}{2\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{e^t\left[e^{\varepsilon t^{1-\alpha}} - 1\right] + e^{-t}\left[1 - e^{\varepsilon t^{1-\alpha}}\right]}{2\varepsilon}$$

$$\left\{\because \sinh t = \frac{e^t - e^{-t}}{2}\right\}$$

$$= \lim_{\varepsilon \to 0} \frac{e^t\left[1 + \varepsilon t^{1-\alpha} + \frac{\left(\varepsilon t^{1-\alpha}\right)^2}{2!} + \ldots - 1\right] + e^{-t}\left[1 - \left(1 - \varepsilon t^{1-\alpha} + \frac{\left(\varepsilon t^{1-\alpha}\right)^2}{2!} - \ldots\right)\right]}{2\varepsilon}$$

{Using the expansions of e^t and e^{-t} }

$$= t^{1-\alpha}\left(\frac{e^t + e^{-t}}{2}\right)$$

$$= t^{1-\alpha} \cosh t$$

$$\left\{\because \frac{e^t + e^{-t}}{2} = \cosh t\right\}$$

(iv) Similarly, $T_\alpha(\cosh t) = t^{1-\alpha} \sinh t$

(v) $T_\alpha(\tan t) = \lim_{\varepsilon \to 0} \frac{\tan\left(t + \varepsilon t^{1-\alpha}\right) - \tan t}{\varepsilon}$

$$= \lim_{\varepsilon \to 0} \frac{\frac{\sin\left(t + \varepsilon t^{1-\alpha}\right)}{\cos\left(t + \varepsilon t^{1-\alpha}\right)} - \frac{\sin t}{\cos t}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\sin\left(t + \varepsilon t^{1-\alpha}\right)\cos t - \sin t \cos\left(t + \varepsilon t^{1-\alpha}\right)}{\varepsilon \cos t \cos\left(t + \varepsilon t^{1-\alpha}\right)}$$

$$= \lim_{\varepsilon \to 0} \frac{\sin\left(t + \varepsilon t^{1-\alpha} - t\right)}{\varepsilon \cos t \cos\left(t + \varepsilon t^{1-\alpha}\right)} \cdot \frac{t^{1-\alpha}}{t^{1-\alpha}}$$

$$= \frac{t^{1-\alpha}}{\cos^2 t} = t^{1-\alpha} \sec^2 t$$

$$\left\{ \lim_{\varepsilon \to 0} \frac{\sin \varepsilon}{\varepsilon} = 1 \right\}$$

Let us write $y^{(n\alpha)}$ to denote the α -derivative of *y*, *n*-times. That is $y^{(n\alpha)} = T_\alpha T_\alpha ... T_\alpha (y)$, *n*-times.

Theorem 1: Let $y^{(n\alpha)} + a_{n-1} y^{(n-1)\alpha} + ... + a_1 y^{(\alpha)} + a_0 y = 0$ (*)

Consider the equation

$$r^{(n\alpha)} + a_{n-1} r^{(n-1)\alpha} + ... + a_1 r^{(\alpha)} + a_0 = 0$$

If $r_1^\alpha = \lambda_1, ..., r_n^\alpha = \lambda_n$ are the real roots of (*) then $y_h = c_1 y_1 + ... + c_n y_n$ where $y_k = e^{r_k^\alpha e^{\frac{t^\alpha}{\alpha}}}$.

a) If one of the roots λ_1 is repeated *k*-times, then

$$y_1, \frac{t^\alpha}{\alpha} y_1, ..., \left(\frac{t^\alpha}{\alpha} \right)^{k-1} y_1, \text{ where } y_1 = e^{\lambda_1 \frac{t^\alpha}{\alpha}} \text{ are independent solutions.}$$

b) If there is a root say, $\lambda_1 = a + ib$ where $a, b \in \square$, then

$$y_1 = e^{a\frac{t^\alpha}{\alpha}} \cos b \frac{t^\alpha}{\alpha}, y_2 = e^{a\frac{t^\alpha}{\alpha}} \sin b \frac{t^\alpha}{\alpha} \text{ are two independent solutions.}$$

2. Simultaneous Conformable Fractional Differential Equations:

Let *x* and *y* be the dependent variables and *t* be the independent variable. Thus, in such equations there occur differential coefficients of *x* and *y* with respect to *t*. Let $D^\alpha = \frac{d^\alpha}{dt^\alpha}$. Then, such equations can be put in the form

$$f_1(D^\alpha) x + f_2(D^\alpha) y = W_1 \quad (1)$$

$$g_1(D^\alpha) x + g_2(D^\alpha) y = W_2 \quad (2)$$

where W_1 and W_2 are functions of the independent variable *t* and $f_1(D^\alpha)$, $f_2(D^\alpha)$, $g_1(D^\alpha)$ and $g_2(D^\alpha)$ are all rational integral functions of D^α with constant coefficients. Such equations can be solved by the following two methods.

First Method: Method of Elimination

In order to eliminate *y* between (1) and (2), operating on both sides of (1) by $g_2(D^\alpha)$ and on both sides of (2) by $f_2(D^\alpha)$ and subtracting, we have

$$\left\{ f_1(D^\alpha) g_2(D^\alpha) - g_1(D^\alpha) f_2(D^\alpha) \right\} x = g_2(D^\alpha) W_1 - f_2(D^\alpha) W_2 \quad (3)$$

which is a conformable linear fractional differential equation with constant coefficients in *x* and *t* and can be solved to give the value of *x* in either (1) or (2), we get the value *y* in terms of *t*. Equation (3) is solved by using methods given in [].

Note 1:

The above equations (1) and (2) can also be solved by first eliminating *x* between them and solving the resulting equation to get *y* in terms of *t*. Substituting this value of *y* in either (1) or (2), we get the value of *x* in terms of *t*.

Second Method: Method of Differentiation:

We can also eliminate *x* and *y* by differentiating (1) and (2). For example, assume that the equations (1) and (2) connect four quantities $x, y, \frac{d^\alpha x}{dt^\alpha}$ and $\frac{d^\alpha y}{dt^\alpha}$. Differentiating (1) and (2) with respect to *t*, we obtain four equations containing $x, \frac{d^\alpha x}{dt^\alpha}, \frac{d^{2\alpha} x}{dt^{2\alpha}}, y, \frac{d^\alpha y}{dt^\alpha}$ and $\frac{d^{2\alpha} y}{dt^{2\alpha}}$. Eliminating three quantities $y, \frac{d^\alpha y}{dt^\alpha}, \frac{d^{2\alpha} y}{dt^{2\alpha}}$ from these four equations, *y* is eliminated and we get an equation of the second order with *x* as the dependent and *t* as the independent variable. Solving this equation we get value of *x* in terms of *t*. Substituting this value of *x* in either (1) or (2), we get value of *y* in terms of *t*.

Example 1: Solve the simultaneous equations

$$\frac{d^\alpha x}{dt^\alpha}-7x+y=0$$

$$\frac{d^\alpha y}{dt^\alpha}-2x-5y=0$$

Solution:

First Method: Method of Elimination

Step 1: Writing D^α for $\frac{d^\alpha}{dt^\alpha}$, the given equations can be rewritten in the symbolic form as follows:

$$\left(D^\alpha-7\right)x+y=0 \tag{1}$$

And $$-2x+\left(D^\alpha-5\right)y=0 \tag{2}$$

Step 2: We now eliminate *x* as follows. Multiplying (1) by 2 and operating (2) by $\left(D^\alpha-7\right)$, we get

$$2\left(D^\alpha-7\right)x+2y=0 \tag{3}$$

$$-2\left(D^\alpha-7\right)x+\left(D^\alpha-7\right)\left(D^\alpha-5\right)y=0 \tag{4}$$

Adding (3) and (4), we get $\left(D^{2\alpha}-12D^\alpha+37\right)y=0$ which is a linear equation with constant coefficients.

Its auxiliary equation is $D^{2\alpha}-12D^\alpha+37=0$ so that $D^\alpha=6\pm i$

$$\therefore \quad y=e^{6\frac{t^\alpha}{\alpha}}\left(c_1\cos\left(\frac{t^\alpha}{\alpha}\right)+c_2\sin\left(\frac{t^\alpha}{\alpha}\right)\right) \tag{5}$$

Where c_1 and c_2 are the arbitrary constants.

Step 3: Now we can get *x* by using (5). Taking the $\alpha-$derivative of (5) w.r.t. *t*, we get

$$D^\alpha y=6e^{6\frac{t^\alpha}{\alpha}}c_1\cos\left(\frac{t^\alpha}{\alpha}\right)-e^{6\frac{t^\alpha}{\alpha}}c_1\sin\left(\frac{t^\alpha}{\alpha}\right)+6e^{6\frac{t^\alpha}{\alpha}}c_2\sin\left(\frac{t^\alpha}{\alpha}\right)+e^{6\frac{t^\alpha}{\alpha}}c_2\cos\left(\frac{t^\alpha}{\alpha}\right)$$

$$D^\alpha y=e^{6\frac{t^\alpha}{\alpha}}\left\{\left(6c_1+c_2\right)\cos\left(\frac{t^\alpha}{\alpha}\right)+\left(6c_2-c_1\right)\sin\left(\frac{t^\alpha}{\alpha}\right)\right\} \tag{6}$$

Substituting the values of *y* and $D^\alpha y$ given in (5) and (6) in (2), we have

$$2x=D^\alpha y-5y=e^{6\frac{t^\alpha}{\alpha}}\left[\left(6c_1+c_2\right)\cos\left(\frac{t^\alpha}{\alpha}\right)+\left(6c_2-c_1\right)\sin\left(\frac{t^\alpha}{\alpha}\right)-5\left(c_1\cos\left(\frac{t^\alpha}{\alpha}\right)+c_2\sin\left(\frac{t^\alpha}{\alpha}\right)\right)\right]$$

$$x=\frac{1}{2}e^{6\frac{t^\alpha}{\alpha}}\left[\left(c_1+c_2\right)\cos\left(\frac{t^\alpha}{\alpha}\right)+\left(c_2-c_1\right)\sin\left(\frac{t^\alpha}{\alpha}\right)\right] \tag{7}$$

Thus, (5) and (7) together give the required solution.

Remark: We can also eliminate *y* first and then obtain *x*. This value of *x* can be put in (1) to get the desired value of *y*.

Second Method: Method of differentiation

$$\frac{d^\alpha x}{dt^\alpha}-7x+y=0 \tag{1}$$

$$\frac{d^\alpha y}{dt^\alpha} - 2x - 5y = 0 \tag{2}$$

To eliminate *x*, we take the $\alpha-$derivative of (2) w.r.t *t* and obtain

$$D^\alpha D^\alpha y - 2D^\alpha x - 5D^\alpha y = 0 \tag{3}$$

Now, from (2), we have

$$x = \frac{1}{2}\left(D^\alpha y - 5y\right) \tag{4}$$

Then, from (1), we get

$$D^\alpha x = 7x - y = \frac{7}{2}\left(D^\alpha y - 5y\right) - y \qquad \{\text{Using (4)}\}$$

$$\therefore \quad D^\alpha x = \frac{7}{2}D^\alpha y - \frac{37}{2}y$$

Substituting this value of $D^\alpha x$ in (3), we have

$$\left(D^{2\alpha} - 12D^\alpha + 37\right)y = 0$$

Now, we can solve for *y* and *x* as done in first method.

Example 2: Solve the simultaneous differential equations

$$\frac{d^\alpha x}{dt^\alpha} = 3x + 2y \tag{1}$$

$$\frac{d^\alpha y}{dt^\alpha} = 5x + 3y \tag{2}$$

Solution: Writing D^α for $\frac{d^\alpha}{dt^\alpha}$, the given equations can be rewritten in the symbolic form as follows:

$$\left(D^\alpha - 3\right)x - 2y = 0 \tag{1}$$

$$-5x + \left(D^\alpha - 3\right)y = 0 \tag{2}$$

Operating on both sides of (1) by $\left(D^\alpha - 3\right)$ and multiplying both sides of (2) by 2 and then adding, we have

$$\left(D^{2\alpha} - 6D^\alpha - 1\right)x = 0 \tag{3}$$

Now, auxiliary equation of (3) is $D^{2\alpha} - 6D^\alpha - 1 = 0$ so that $D^\alpha = 3 \pm \sqrt{10}$

$$\therefore \quad x = C.F. = e^{3\frac{t^\alpha}{\alpha}}\left[c_1 \cosh\left(\frac{t^\alpha}{\alpha}\sqrt{10}\right) + c_2 \sinh\left(\frac{t^\alpha}{\alpha}\sqrt{10}\right)\right] \tag{4}$$

From (4),

$$D^\alpha x = e^{3\frac{t^\alpha}{\alpha}}\left[\left(3c_1 + c_2\sqrt{10}\right)\cosh\left(\frac{t^\alpha}{\alpha}\sqrt{10}\right) + \left(3c_2 + c_1\sqrt{10}\right)\sinh\left(\frac{t^\alpha}{\alpha}\sqrt{10}\right)\right]$$

Then, from (1), we have

$$y = \frac{1}{2}\left(D^\alpha - 3\right)x$$

$$= \frac{\sqrt{10}}{2}e^{3\frac{t^\alpha}{\alpha}}\left[c_2 \cosh\left(\frac{t^\alpha}{\alpha}\sqrt{10}\right) + c_1 \sinh\left(\frac{t^\alpha}{\alpha}\sqrt{10}\right)\right]$$

Conclusion:

Conformable fractional derivatives can be applied to solve simultaneous linear differential equation with constant coefficients.

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