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CUBIC IDEALS IN SUBTRACTION SEMIGROUPS

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Abstract:

In this paper, we introduce the notion of cubic ideals in subtraction semigroups and study several related properties are investigated. Some results are obtained.

Index Terms: Subtraction Algebra, Sub Algebra, Subtraction Semigroup, Sub-Subtraction Semigroup, An Ideal, Cubic Subtraction Semigroup, Cubic Ideals & Cubic Homomorphisam.

1. Introduction:

The system of the form $(\varphi, \circ, \setminus)$. Here φ is a set of function closed under the composition "∘" of function (and hence $(\varphi, \circ)$ is a function of semigroup) and the set theoretical subtraction "\"(*and hence* $(\varphi, \setminus)$ is subtraction algebra) in the sense of [1]. Schein [10] introduced the concept of subtraction semigroup in 1992. Solved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Zelinka [14] discussed a problem proposed by Schein [10] concerning the structure of multiplication in a subtraction semigroup and discussed a special type of subtraction algebra denoted atomic subtraction algebra. Jun et al [8] introduced the notion of ideals in subtraction algebra and discussed and discussed characterization of ideals. Dheena et al [4] discussed the fundamental properties related to ideals and sub subtraction semigroups. Jun et al [9] introduced the concept of cubic sets. This structure encompasses interval-valued fuzzy set and fuzzy set. Also Jun et al [7] introduced the notion of cubic subgroups. Vijayabalaji et al [11] introduced the notion of cubic linear space. Chinnadurai et al [3] introduced the notion of cubic ring. The concept of fuzzy subset was introduced by Zadeh [12, 13] in order to study mathematical vague situations. Many researchers who are involved in studying, applying, refining and teaching fuzzy sets have successfully applied this theory in many different fields. The purpose of this paper to introduce the notion of cubic ideals in subtraction semigroups and homomorphism of subtraction semigroups. We investigate some basic results, examples and properties.

2. Preliminaries:

Now we recall some known concepts related to cubic ideals in subtraction semigroups from the literature, which will be needed in the sequel.

Definition 2.1 [1] A non-empty set X together with a binary operation "-" is said to be a subtraction algebra if it satisfies the following assertions holds,

i) $x - (y - x) = x$
ii) $x - (x - y) = y - (y - x)$
iii) $(x - y) - z = (x - z) - y \ \forall x, y, z \in X$.

Definition 2.2 [1] Let A be any non-empty set. Then (P (A), \) is subtraction algebra, where P (A) denotes the power set of A and "\" denotes the set theoretic subtraction.

Definition 2.3 [1] A subset I of subtraction algebra X is called subalgebra of X if $x - y \in I$ for all $x, y \in X$.

In subtraction algebra the following holds: [1]

S1) $x - 0 = x$ and $0 - x = 0$
S2) $x - (x - y) \le y$
S3) $x \le y$ if and only if $x = y - w$ for some $w \in X$
S4) $x \le y$ implies $x - z \le y - z$ and $z - y \le z - x$ for all $z \in X$
S5) $x - (x - (x - y)) = x - y$
S6) $(x - y) - x = 0$
S7) $(x - y) - y = x - y$.

Definition2.4 [1] Anon-empty set X together with the binary operation "-" and "." is said to be a subtraction semigroup. If it satisfies the following conditions

i) (X,-) is a subtraction algebra
ii) (X, .) is a semigroup
iii) $x(y - z) = xy - xz$ and $(x - y)z = xz - yz \ \forall x, y, z \in X$.

Definition 2.5 [1] Anon-empty subset S of a subtraction semigroup X is said to be sub subtraction semigroup of X. If it satisfies the following conditions

i) $x - y \in S$
ii) $xy \in S \ \ \forall x, y \in X$.

Definition 2.6 [1] A non-empty subset I of a subtraction semigroup X is said to be a left (right) ideal, if

i) $y \in I$ and $x - y \in I$ imply $x \in I$ for all $x, y \in X$;

ii) XI ⊆ I, (IX ⊆ I).
If I is both a left and right ideal, it is called a two –sided ideal (simply, ideal) of X.
Definition 2.7 [2] A mapping $\mu: X \to [0,1]$ is called a fuzzy subset of X.
Definition 2.8 [2] A fuzzy subset $\mu$of X is called fuzzy sub subtraction semigroup of X, if
$i)\ \mu(x-y) \geq \min\{\mu(x), \mu(y)\}$
$ii)\ \mu(xy) \geq \min\{\mu(x), \mu(y)\} \quad \forall x, y \in X.$
Definition 2.9 [2] A fuzzy subset μ of X is called fuzzy left (right) ideal of X, if
$i)\ \mu(x) \geq \min\{\mu(x-y), \mu(y)\}$
$ii)\ \mu(xy) \geq \min\{\mu(x), \mu(y)\}$
$iii)\ \mu(xy) \geq \mu(y)\big(\mu(xy) \geq \mu(x)\big) \quad \forall x, y \in X.$
Definition 2.10 [5] Let X be a non-empty set. A mapping $\bar{\mu}: X \to D[0,1]$ is called interval-valued fuzzy set (in short i-v), where D[0,1] denote the family of all closed sub intervals of [0,1] and $\bar{\mu}(x) = [\mu^-(x), \mu^+(x)]$ for all $x \in X$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$ for all $x \in X$.
Definition 2.11 [7] Let X be a non-empty set. A cubic set $\mathcal{A}$ in X is a structure of the form
$\mathcal{A}=\{\langle x, \bar{\mu}_A(x), \lambda(x)\rangle : x \in X\}$ and denoted by $\mathcal{A}=\langle \bar{\mu}_A, \lambda\rangle$, where $\bar{\mu}_A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set (briefly, IVF) in X and λ is a fuzzy set in X.
Definition 2.12 [9] The complement of $\mathcal{A}=\langle \bar{\mu}_A, \gamma\rangle$ is defined to be the cubic set
$\mathcal{A}^C = \{\langle x, (\bar{\mu}_A)^C(x), 1-\gamma(x)\rangle | x \in X\}$.
Definition 2.13 [9] For any $\mathcal{A}_i = \{\langle x, \bar{\mu}_i(x), \lambda_i(x)\rangle | x \in X\}$ where i ∈⋀ (index set), we have the following,
$i)\ \cap_{R, i\in\wedge} \mathcal{A}_i = \{\langle x, (\cap_{i\in\wedge} \bar{\mu}_i)(x), (\cup_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\}, (R-intersection)$
$ii)\ \cup_{R, i\in\wedge} \mathcal{A}_i = \{\langle x, (\cup_{i\in\wedge} \bar{\mu}_i)(x), (\cap_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\}, (R-union)$
$iii)\ \cap_{P, i\in\wedge} \mathcal{A}_i = \{\langle x, (\cap_{i\in\wedge} \bar{\mu}_i)(x), (\cap_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\}, (P-intersection)$
$iv)\ \cup_{P, i\in\wedge} \mathcal{A}_i = \{\langle x, (\cup_{i\in\wedge} \bar{\mu}_i)(x), (\cup_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\}, (P-union)$
Definition 2.14 [11] Let$\mathcal{A} = <\bar{\mu}, \gamma>$ be a cubic set of S. Define
$U(\mathcal{A}\,; \tilde{t}, n) = \{x \in S | \bar{\mu}(x) \geq \tilde{t} \text{ and } \gamma(x) \leq n\}$ where $\tilde{t} \in D[0,1]$ and $n \in [0,1]$ is called the cubic level set of $\mathcal{A}$.
Definition 2.15 [6] For any non-empty subset G of a set X, the characteristic cubic set of G is defined to be a structure $\chi_G(x) =< x, \bar{\mu}_{\chi_G}(x), \gamma_{\chi_G}(x) : x \in X >$ which is briefly denoted by $\chi_G(x) =< \bar{\mu}_{\chi_G}(x), \gamma_{\chi_G}(x) >$ where
$\bar{\mu}_{\chi_G}(x) = \begin{cases} [1,1] & if\ x \in G \\ [0,0] & otherwise \end{cases}$ and $\gamma_{\chi_G}(x) = \begin{cases} 0 & if\ x \in G \\ 1 & otherwise \end{cases}$

3. Cubic Ideals in Subtraction Semigroups:

In this section, we introduced the notion of cubic ideals in subtraction semigroups and discuss some of its properties.

Definition 3.1 Let S be a subtraction semigroup, $(S, \bar{\mu})$ be an interval-valued fuzzy sub subtraction semigroup and (S, γ) be a fuzzy sub subtraction semigroup. A cubic set $\mathcal{A} = \langle \bar{\mu}, \gamma\rangle$ is called a cubic sub subtraction semigroup of S. If it satisfies the following conditions
$i)\ \bar{\mu}(x-y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$ii)\ \gamma(x-y) \leq \max\{\gamma(x), \gamma(y)\}$
$iii)\ \bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$iv)\ \gamma(xy) \leq \max\{\gamma(x), \gamma(y)\} \quad for\ all\ x, y \in S.$
Example 3.2 Let S = {0, a, b, 1} in which "-" and "." are defined as

-	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

·	0	a	b	1
0	0	0	0	0
a	0	a	0	0
b	0	0	b	b
1	0	0	b	b

Define an interval-valued fuzzy set $\bar{\mu}$:S → D[0,1] by $\bar{\mu}$ (0)=[0.9,1], $\bar{\mu}$ (a)=[0.6,0.7], $\bar{\mu}$ (b)=[0.8,0.9] and $\bar{\mu}$(1)=[0,0.1] is an interval-valued fuzzy sub- subtraction semigroup of S. Define a fuzzy set γ:S→[0,1] by γ(0)=0, γ (a)=0.67, γ (b)=0.72 and γ(1)=1 is a fuzzy sub- subtraction semigroup of S. Hence $\mathcal{A} = <\bar{\mu}, \gamma>$ is a cubic sub-subtraction semigroup of S.
Definition 3.3 A cubic set $\mathcal{A} = \langle \bar{\mu}, \gamma\rangle$ of S is called a cubic left (right) ideal of S. If it satisfies the following conditions
$i)\ \bar{\mu}(x) \geq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}$
$ii)\ \gamma(x) \leq \max\{\gamma(x-y), \gamma(y)\}$
$iii)\ \bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$iv)\ \gamma(xy) \leq \max\{\gamma(x), \gamma(y)\}$
$v)\ \bar{\mu}(xy) \geq \bar{\mu}(y), [\bar{\mu}(xy) \geq \bar{\mu}(x)]$

vi) $\gamma(xy) \leq \gamma(y), [\gamma(xy) \leq \gamma(x)] \quad \forall x, y \in S.$

Example 3.4 Let S = {0, a, b, 1} in which "-" and "." are defined as

-	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

·	0	a	b	1
0	0	0	0	0
a	0	a	0	0
b	0	0	b	b
1	0	0	b	b

Define an interval-valued fuzzy set $\bar{\mu}$:S → D[0,1] by $\bar{\mu}$ (0)=[0.9,1], $\bar{\mu}$ (a)=[0.5,0.6], $\bar{\mu}$ (b)=[0.7,0.8] and $\bar{\mu}$(1)=[0.1,0.2] is an interval-valued fuzzy left (right) ideal of S. Define a fuzzy set γ:S→[0,1] by γ(0)=0, γ(a)=0.25, γ(b)=0.5 and γ(1)=1is a fuzzy left (right) ideal of S. Hence $\mathcal{A} = <\bar{\mu}, \gamma>$ is a cubic left (right) ideal of subtraction semigroup S.

Definition 3.5 *Let* $\mathcal{A}_1 =< \bar{\mu}_1, \gamma_1 >$ and $\mathcal{A}_2 =< \bar{\mu}_2, \gamma_2 >$ be any two cubic sets of S then the following cubic sets of S are defined as follows,

$$(\mathcal{A}_1 - \mathcal{A}_2)(z) = \begin{cases} (\bar{\mu}_1 - \bar{\mu}_2)(z) = \begin{cases} \sup\limits_{z=x-y} \min\{\bar{\mu}_1(x), \bar{\mu}_2(y)\}, \ \forall x, y \in S \\ [0,0] \quad otherwise \end{cases} \\ (\gamma_1 - \gamma_2)(z) = \begin{cases} \inf\limits_{z=x-y} \max\{\gamma_1(x), \gamma_2(y)\}, \ \forall x, y \in S \\ 1 \quad otherwise \end{cases} \end{cases}$$

$$(\mathcal{A}_1 * \mathcal{A}_2)(x) = \begin{cases} (\bar{\mu}_1 * \bar{\mu}_2)(x) = \begin{cases} \sup\limits_{x \leq ab} \min\{\bar{\mu}_1(a), \bar{\mu}_2(b)\} \ if\ x \leq ab \\ [0,0] \quad otherwise \end{cases} \\ (\gamma_1 * \gamma_2)(x) = \begin{cases} \inf\limits_{x \leq ab} \max\{\gamma_1(a), \gamma_2(b)\} \ if\ x \leq ab \\ 1 \quad otherwise \end{cases} \end{cases}$$

$$(\mathcal{A}_1 \cap \mathcal{A}_2)(x) = \begin{cases} (\bar{\mu}_1 \cap \bar{\mu}_2)(x) \\ (\gamma_1 \cup \gamma_2)(x) \end{cases}$$

Lemma 3.6

Let $\mathcal{A} = <\bar{\mu}, \gamma>$ be a cubic left (right) ideal of S, then $\bar{\mu}(0) \geq \bar{\mu}(x)$ and $\gamma(0) \leq \gamma(x)\ \forall x \in S.$

Proof:

Let $x \in S$ then $\bar{\mu}(0) = \bar{\mu}(0.x) \geq \bar{\mu}(x)$ thus $\bar{\mu}(0) \geq \bar{\mu}(x)$ and
$\gamma(0) = \gamma(0.x) \leq \gamma(x)$ thus $\gamma(0) \leq \gamma(x)$.

Theorem 3.7:

Every cubic left (right) ideal of S is a cubic sub subtraction semigroup of S.

Proof:

Let $\mathcal{A} = \langle\bar{\mu}, \gamma\rangle$ be a cubic left (right) ideal of S.

Since $\bar{\mu}$ be an interval-valued fuzzy left ideal of S.

$\bar{\mu}(x-y) \geq \min\{\bar{\mu}((x-y)-z), \bar{\mu}(z)\}\ \ \forall z \in S$
$\geq \min\{\bar{\mu}((x-y)-x), \bar{\mu}(x)\}\ \ for\ z = x$
$\geq \min\{\bar{\mu}(0), \bar{\mu}(x)\}$ *(by lemma)*
$\bar{\mu}(x-y) \geq \bar{\mu}(x)$ again consider
$\bar{\mu}(x-y) \geq \min\{\bar{\mu}((x-y)-z), \bar{\mu}(z)\}\ \ \forall z \in S$
$\geq \min\{\bar{\mu}((x-y)-y), \bar{\mu}(y)\}\ \ for\ z = y$
$\geq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}$
$\bar{\mu}(x-y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
Clearly $\bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
Since γ is a fuzzy left ideal of S,
$\gamma(x-y) \leq \max\{\gamma((x-y)-z), \gamma(z)\}\ \forall z \in S$
$\leq \max\{\gamma((x-y)-x), \gamma(x)\}\ for\ z = x$
$\leq \max\{\gamma(0), \gamma(x)\}$ (by lemma)
$\gamma(x-y) \leq \gamma(x)$, again consider
$\gamma(x-y) \leq \max\{\gamma((x-y)-z), \gamma(z)\}\ \forall z \in S$
$\leq \max\{\gamma((x-y)-y), \gamma(y)\}\ for\ z = y$
$\leq \max\{\gamma(x-y), \gamma(y)\}$
$\gamma(x-y) \leq \max\{\gamma(x), \gamma(y)\}$
Clearly $\gamma(xy) \leq \max\{\gamma(x), \gamma(y)\}$
Thus $\mathcal{A} = \langle\bar{\mu}, \gamma\rangle$ is a cubic sub-subtraction semigroup of S.

Theorem 3.8:

Let $\mathcal{A} = <\bar{\mu}, \gamma>$ be a cubic sub-subtraction semigroup (cubic ideal) of S then $\mathcal{A} - \mathcal{A} = \mathcal{A}$.

Proof:

Let $\bar{\mu}$ be an interval-valued fuzzy sub-subtraction semigroup of S and z ∈ S, then
$(\bar{\mu}-\bar{\mu})(z) = \sup_{z=x-y} \min\{\bar{\mu}(x), \bar{\mu}(y)\}, \quad \forall x, y \in S$
$\geq \min\{\bar{\mu}(z), \bar{\mu}(0)\}$, (by lemma)
$(\bar{\mu}-\bar{\mu})(z) \geq \bar{\mu}(z)$
On the other hand if z = x-y
$\bar{\mu}(z) = \bar{\mu}(x-y)$
$\geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$\geq \sup_{z=x-y} \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$\bar{\mu}(z) \geq (\bar{\mu}-\bar{\mu})(z)$
This implies that $(\bar{\mu}-\bar{\mu})(z) = \bar{\mu}(z)$ then $(\bar{\mu}-\bar{\mu}) = \bar{\mu}$.
Let γ be a fuzzy sub-subtraction semigroup of S and z ∈ S then
$(\gamma-\gamma)(z) = \inf_{z=x-y} \max\{\gamma(x), \gamma(y)\}, \forall x, y \in S$
$\leq \max\{\gamma(z), \gamma(0)\}$ (by lemma)
$(\gamma-\gamma)(z) \leq \gamma(z)$.
On the other hand if z = x-y
$\gamma(z) = \gamma(x-y)$
$\leq \max\{\gamma(x), \gamma(y)\}$
$\leq \inf_{z=x-y} \max\{\gamma(x), \gamma(y)\}$
$\gamma(z) \leq (\gamma-\gamma)(z)$
This implies that $\gamma(z) = (\gamma-\gamma)(z)$ then $\gamma = (\gamma-\gamma)$
Hence $\mathcal{A} - \mathcal{A} = \mathcal{A}$.

Theorem 3.9:

Let $\mathcal{A} = <\bar{\mu}, \gamma>$ be a cubic subset of S. If $\mathcal{A}$ is a cubic left (resp. right) ideal of S then the set $S_{\mathcal{A}} = \{x \in S \mid \mathcal{A}(x) = \mathcal{A}(0)\}$ is an ideal of S.

Proof:

Let $\mathcal{A}$ be a cubic ideal of S and x, y ∈ S then $\mathcal{A}(x) = \mathcal{A}(0)$ and $\mathcal{A}(y) = \mathcal{A}(0)$.
Suppose $x-y, y \in S_{\mathcal{A}}$ then $\bar{\mu}(x-y) = \bar{\mu}(0)$; $\bar{\mu}(y) = \bar{\mu}(0)$ and $\gamma(x-y) = \gamma(0)$; $\gamma(y) = \gamma(0)$
$\bar{\mu}(x) \geq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\} = \min\{\bar{\mu}(0), \bar{\mu}(0)\} = \bar{\mu}(0)$ and
$\gamma(x) \leq \max\{\gamma(x-y), \gamma(y)\} = \max\{\gamma(0), \gamma(0)\} = \gamma(0)$ this implies that $x \in S_{\mathcal{A}}$
Suppose $x, y \in S_{\mathcal{A}}$ then $\bar{\mu}(x) = \bar{\mu}(0)$; $\bar{\mu}(y) = \bar{\mu}(0)$ and $\gamma(x) = \gamma(0)$; $\gamma(y) = \gamma(0)$.
$\bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} = \min\{\bar{\mu}(0), \bar{\mu}(0)\} = \bar{\mu}(0)$ and
$\gamma(xy) \leq \max\{\gamma(x), \gamma(y)\} = \max\{\gamma(0), \gamma(0)\} = \gamma(0)$ thus $xy \in S_{\mathcal{A}}$
For every $x \in S$ and $y \in S_{\mathcal{A}}$ $\bar{\mu}(xy) \geq \bar{\mu}(y) = \bar{\mu}(0)$ and $\gamma(xy) \leq \gamma(y) = \gamma(0)$ this implies that $xy \in S_{\mathcal{A}}$
Hence $S_{\mathcal{A}}$ is an ideal of S.

Theorem 3.10:

If $\mathcal{A} = <\bar{\mu}, \gamma>$ is a cubic sub-subtraction semigroup of S, then the following are equivalent
$i)\ \bar{\mu} * \bar{\mu} \leq \bar{\mu}$ *and* $\gamma * \gamma \geq \gamma$
$ii)\ \bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ *and* $\gamma(xy) \leq \max\{\gamma(x), \gamma(y)\}\ \forall x, y \in S$.

Proof:

Let x, y ∈ S. $i) \to ii)$
Consider $(\bar{\mu} * \bar{\mu})(xy) = \sup_{xy \leq ab} \min\{\bar{\mu}(a), \bar{\mu}(b)\} \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$. By (i) $\bar{\mu} * \bar{\mu} \leq \bar{\mu}$
$\bar{\mu}(xy) \geq (\bar{\mu} * \bar{\mu})(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$.
Hence $\bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$.
$(\gamma * \gamma)(xy) = \inf_{xy \leq ab} \max\{\gamma(a), \gamma(b)\} \leq \max\{\gamma(x), \gamma(y)\}$. By (i) $\gamma * \gamma \geq \gamma$
$\gamma(xy) \leq (\gamma * \gamma)(xy) \leq \max\{\gamma(x), \gamma(y)\}$.
Hence $\gamma(xy) \leq \max\{\gamma(x), \gamma(y)\}$.
$ii) \to i)$ Let $x \in S$. Consider $(\bar{\mu} * \bar{\mu})(x) = \sup_{x \leq ab} \min\{\bar{\mu}(a), \bar{\mu}(b)\} \leq \sup_{x \leq ab} \bar{\mu}(ab) \leq \bar{\mu}(x)$
Thus $\bar{\mu} * \bar{\mu} \leq \bar{\mu}$
if x cannot be expressed as $x \leq ab$, then $(\bar{\mu} * \bar{\mu})(x) = \bar{0} \leq \bar{\mu}(x)$ this implies that
$(\bar{\mu} * \bar{\mu})(x) \leq \bar{\mu}(x)$. Hence $\bar{\mu} * \bar{\mu} \leq \bar{\mu}$ and
$(\gamma * \gamma)(x) = \inf_{x \leq ab} \max\{\gamma(a), \gamma(b)\} \geq \inf_{x \leq ab} \gamma(ab) \geq \gamma(x)$
Thus $\gamma * \gamma \geq \gamma$
If x cannot be expressed as $x \leq ab$ then $(\gamma * \gamma)(x) = 1 \geq \gamma(x)$ this implies that $(\gamma * \gamma)(x) \geq \gamma(x)$.
Hence $\gamma * \gamma \geq \gamma$.

Theorem 3.11:

If $\{\mathcal{A}_i\} = <\bar{\mu}_i, \gamma_i | i \in \wedge>$ is a family of cubic left (right) ideals of S then $\sqcap_{i\in\wedge}\mathcal{A}_i = <\cap_{i\in\wedge}\bar{\mu}_i, \cup_{i\in\wedge}\gamma_i>$ is also a cubic left (right) ideal of S, where $\wedge$ is any index set.

Proof:

Let $\mathcal{A}_i = <\bar{\mu}_i, \gamma_i | i \in \wedge>$ be a family of cubic left (right) ideals of S.

Let x, y $\in S$ and $\bar{\mu} = \cap\bar{\mu}_i$; $\gamma = \cup\gamma_i$ then

$\bar{\mu}(x) = \cap\bar{\mu}_i(x) = (inf\ \bar{\mu}_i)(x) = \inf\bar{\mu}_i(x)$ and $\gamma(x) = \cup\gamma_i(x) = (sup\ \gamma_i)(x) = \sup\gamma_i(x)$

$i)\ \bar{\mu}(x) = \inf\bar{\mu}_i(x)$
$\geq \inf\ \min\{\bar{\mu}_i(x-y), \bar{\mu}_i(y)\}$
$= \min\{inf\ \bar{\mu}_i(x-y), inf\ \bar{\mu}_i(y)\}$
$= \min\{\cap\bar{\mu}_i(x-y), \cap\bar{\mu}_i(y)\}$
$\bar{\mu}(x) \geq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}$

$ii)\ \gamma(x) = sup\ \gamma_i(x)$
$\leq sup\ \max\{\gamma_i(x-y), \gamma_i(y)\}$
$= \max\{sup\ \gamma_i(x-y),\ sup\ \gamma_i(y)\}$
$= \max\{\cup\gamma_i(x-y), \cup\gamma_i(y)\}$
$\gamma(x) \leq \max\{\gamma(x-y), \gamma(y)\}$

$iii)\ \bar{\mu}(xy) = \inf\bar{\mu}_i(xy)$
$\geq \inf\ \min\{\bar{\mu}_i(x), \bar{\mu}_i(y)\}$
$= \min\{inf\ \bar{\mu}_i(x), inf\ \bar{\mu}_i(y)\}$
$= \min\{\cap\bar{\mu}_i(x), \cap\bar{\mu}_i(y)\}$
$\bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$

$iv)\ \gamma(xy) = sup\ \gamma_i(xy)$
$\leq sup\ \max\{\gamma_i(x), \gamma_i(y)\}$
$= \max\{sup\ \gamma_i(x),\ sup\ \gamma_i(y)\}$
$= \max\{\cup\gamma_i(x), \cup\gamma_i(y)\}$
$\gamma(xy) \leq \max\{\gamma(x), \gamma(y)\}$

$v)\ \bar{\mu}(xy) = \inf\bar{\mu}_i(xy) \geq \inf\bar{\mu}_i(y) \geq \bar{\mu}(y)$

$vi)\ \gamma(xy) = sup\ \gamma_i(xy) \leq sup\ \gamma_i(y) \leq \gamma(y)$

Hence $\sqcap_{i\in\wedge}\mathcal{A}_i = <\cap_{i\in\wedge}\bar{\mu}_i, \cup_{i\in\wedge}\gamma_i>$ is a cubic left (right) ideal of S.

Theorem 3.12:

If $\mathcal{A} = <\bar{\mu}, \gamma>$ be any cubic set of S then $\mathcal{A} = <\bar{\mu}, \gamma>$ is a cubic left (right) ideal of S if and only if the cubic level set $U(\mathcal{A}; \tilde{t}, n)$ is a left (right) ideal of S when it is non-empty.

Proof:

Assume that $\mathcal{A} = <\bar{\mu}, \gamma>$ be a cubic left (right) ideal of S.

Let $x, y, x-y \in U(\mathcal{A}; \tilde{t}, n)$ for all $\tilde{t} \in D[0,1]$and $n \in [0,1]$. Then

$\bar{\mu}(x) \geq \tilde{t}, \bar{\mu}(y) \geq \tilde{t}, \bar{\mu}(x-y) \geq \tilde{t}$ and $\gamma(x) \leq n, \gamma(y) \leq n, \gamma(x-y) \leq n.$

Now suppose $x-y, y \in U(\mathcal{A}; \tilde{t}, n)$ then $\bar{\mu}(x) \geq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\} \geq \min\{\tilde{t}, \tilde{t}\} \geq \tilde{t}$ and $\gamma(x) \leq \max\{\gamma(x-y), \gamma(y)\} \leq \max\{n, n\} \leq n$. Hence $x \in U(\mathcal{A}; \tilde{t}, n)$.

Suppose $x, y \in U(\mathcal{A}; \tilde{t}, n)$ then $\bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} \geq \min\{\tilde{t}, \tilde{t}\} \geq \tilde{t}$ and $\gamma(xy) \leq \max\{\gamma(x), \gamma(y)\} \leq \max\{n, n\} \leq n$. Hence $xy \in U(\mathcal{A}; \tilde{t}, n)$.

Let $x \in S$ and $y \in U(\mathcal{A}; \tilde{t}, n)$ then $\bar{\mu}(xy) \geq \bar{\mu}(y) \geq \tilde{t}$ and $\gamma(xy) \leq \gamma(y) \leq n$.

This implies that $xy \in U(\mathcal{A}; \tilde{t}, n)$. Hence $U(\mathcal{A}; \tilde{t}, n)$ is a left (right) ideal of S.

Conversely, let $\tilde{t} \in D[0,1]$ and $n \in [0,1]$ be such that $U(\mathcal{A}; \tilde{t}, n) \neq \emptyset$ and $U(\mathcal{A}; \tilde{t}, n)$ is a left (right) ideal of S.

Suppose we assume that $\bar{\mu}(x) \ngeq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}$ (or) $\gamma(x) \nleq \max\{\gamma(x-y), \gamma(y)\}$.

If $\bar{\mu}(x) \ngeq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}$, then there exist $\tilde{t}_1 \in D[0,1]$ such that $\bar{\mu}(x) < \tilde{t}_1 < \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}$ hence $x-y, y \in U(\mathcal{A}; \tilde{t}_1, \max\{\gamma(x-y), \gamma(y)\})$ but $x \notin U(\mathcal{A}; \tilde{t}_1, \max\{\gamma(x-y), \gamma(y)\})$ which is a contradiction.

If $\gamma(x) \nleq \max\{\gamma(x-y), \gamma(y)\}$ then there exist $n_1 \in [0,1]$ such that $\gamma(x) > n_1 > \max\{\gamma(x-y), \gamma(y)\}$ hence x$-y, y \in U(\mathcal{A}; \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}, n_1)$ but $x \notin U(\mathcal{A}; \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}, n_1)$. This gives a contradiction. Hence $\bar{\mu}(x) \geq \min\{\bar{\mu}(x-y), \bar{\mu}(y)\}$ and $\gamma(x) \leq \max\{\gamma(x-y), \gamma(y)\}$.

Let us assume that $\bar{\mu}(xy) \ngeq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ (or) $\gamma(xy) \nleq \max\{\gamma(x), \gamma(y)\}$.

If $\bar{\mu}(xy) \ngeq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$, then there exist $\tilde{t}_1 \in D[0,1]$ such that $\bar{\mu}(xy) < \tilde{t}_1 < \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ hence $x, y \in U(\mathcal{A}; \tilde{t}_1, \max\{\gamma(x), \gamma(y)\})$ but $xy \notin U(\mathcal{A}; \tilde{t}_1, \max\{\gamma(x), \gamma(y)\})$ which is a contradiction.

If $\gamma(xy) \nleq \max\{\gamma(x), \gamma(y)\}$, then there exist $n_1 \in [0,1]$ such that $\gamma(xy) > n_1 > \max\{\gamma(x), \gamma(y)\}$ hence $x, y \in U(\mathcal{A}; \min\{\bar{\mu}(x), \bar{\mu}(y)\}, n_1)$, but $xy \notin U(\mathcal{A}; \min\{\bar{\mu}(x), \bar{\mu}(y)\}; n_1)$. This gives a contradiction.

Hence $\bar{\mu}(xy) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $\gamma(xy) \leq \max\{\gamma(x), \gamma(y)\}$.

iii) Assume that $\bar{\mu}(xy) \ngeq \bar{\mu}(y)$ (or) $\gamma(xy) \nleq \gamma(y)$. If $\bar{\mu}(xy) \ngeq \bar{\mu}(y)$ then there exist $\tilde{t}_1 \in D[0,1]$such that $\bar{\mu}(xy) < \tilde{t}_1 < \bar{\mu}(y)$ hence $y \in U(\mathcal{A}; \tilde{t}_1, \gamma(y))$ but $xy \notin U(\mathcal{A}; \tilde{t}_1, \gamma(y))$which is a contradiction.

If $\gamma(xy) \nleq \gamma(y)$ then there exist $n_1 \in [0,1]$ such that $\gamma(xy) > n_1 > \gamma(y)$ hence $y \in U(\mathcal{A}\,; \bar{\mu}(y), n_1)$ but $xy \notin U(\mathcal{A}\,; \bar{\mu}(y), n_1)$ which is a contradiction. Hence $\bar{\mu}(xy) \geq \bar{\mu}(y)$ and $\gamma(xy) \leq \gamma(y)$.
Therefore $\mathcal{A} =< \bar{\mu}, \gamma >$ is a cubic left (right) ideal of S.

Theorem 3.13:

Let H be a non-empty subset of S. Then H is a left (right) ideal of S if and only if the characteristic cubic set $\chi_H =< \bar{\mu}_{\chi_H}, \gamma_{\chi_H} >$ of H in S is a cubic left (right) ideal of S.

Proof:

Assume that H is a left (right) ideal of S. Let $x, y \in S$. Suppose that
$\bar{\mu}_{\chi_H}(x) < \min\{\bar{\mu}_{\chi_H}(x-y), \bar{\mu}_{\chi_H}(y)\}$ and $\gamma_{\chi_H}(x) > \max\{\gamma_{\chi_H}(x-y), \gamma_{\chi_H}(y)\}$. It follows that
$\bar{\mu}_{\chi_H}(x) = \bar{0}$, $\min\{\bar{\mu}_{\chi_H}(x-y), \bar{\mu}_{\chi_H}(y)\} = \bar{1}$ and $\gamma_{\chi_H}(x) = 1, \max\{\gamma_{\chi_H}(x-y), \gamma_{\chi_H}(y)\} = 0$.
This implies that $x - y, y \in H$ but $x \notin H$ a contradiction to H being a subtraction semigroup of S.
Suppose that $\bar{\mu}_{\chi_H}(xy) < \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\}$ and $\gamma_{\chi_H}(xy) > \max\{\gamma_{\chi_H}(x), \gamma_{\chi_H}(y)\}$.
It follows that $\bar{\mu}_{\chi_H}(xy) = \bar{0}$, $\min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\} = \bar{1}$ and $\gamma_{\chi_H}(xy) = 1$,
$\max\{\gamma_{\chi_H}(x-y), \gamma_{\chi_H}(y)\} = 0$. This implies that $x, y \in H$ but $xy \notin H$ a contradiction to H.
Suppose that $\bar{\mu}_{\chi_H}(xy) < \bar{\mu}_{\chi_H}(y)$ and $\gamma_{\chi_H}(xy) > \gamma_{\chi_H}(y)$. It follows that $\bar{\mu}_{\chi_H}(xy) = \bar{0}, \bar{\mu}_{\chi_H}(y) = \bar{1}$ and $\gamma_{\chi_H}(xy) = 1, \gamma_{\chi_H}(y) = 0$. This implies that $y \in H$ but $xy \notin H$ a contradiction to H.
Hence $\chi_H =< \bar{\mu}_{\chi_H}, \gamma_{\chi_H} >$ is a cubic left (right) ideal of S.
Conversely, assume that $\chi_H =< \bar{\mu}_{\chi_H}, \gamma_{\chi_H} >$ is a cubic left (right) ideal of S, for any subset H of S.
Let $x - y, y \in H$ for any $x, y \in S$, then $\bar{\mu}_{\chi_H}(x-y) = \bar{\mu}_{\chi_H}(y) = \bar{1}$ and
$\gamma_{\chi_H}(x-y) = \gamma_{\chi_H}(y) = 0$, since χ_H is a cubic left (right) ideal of S.
$\bar{\mu}_{\chi_H}(x) \geq \min\{\bar{\mu}_{\chi_H}(x-y), \bar{\mu}_{\chi_H}(y)\} \geq \min\{\bar{1}, \bar{1}\} \geq \bar{1}$ and
$\gamma_{\chi_H}(x) \leq \max\{\gamma_{\chi_H}(x-y), \gamma_{\chi_H}(y)\} \leq \max\{0,0\} \leq 0$. This implies that $x \in H$.
Let $x, y \in H$ then $\bar{\mu}_{\chi_H}(x) = \bar{\mu}_{\chi_H}(y) = \bar{1}$ and $\gamma_{\chi_H}(x) = \gamma_{\chi_H}(y) = 0$
$\bar{\mu}_{\chi_H}(xy) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\} \geq \min\{\bar{1}, \bar{1}\} \geq \bar{1}$ and
$\gamma_{\chi_H}(xy) \leq \max\{\gamma_{\chi_H}(x), \gamma_{\chi_H}(y)\} \leq \max\{0,0\} \leq 0$. Which implies that $xy \in H$.
Let $x \in S$ and $y \in H$ then $\bar{\mu}_{\chi_H}(y) = \bar{1}$ and $\gamma_{\chi_H}(y) = 0$,
$\bar{\mu}_{\chi_H}(xy) \geq \bar{\mu}_{\chi_H}(y) \geq \bar{1}$ and $\gamma_{\chi_H}(xy) \leq \gamma_{\chi_H}(y) \leq 0$. This gives that $xy \in H$.
Hence H is a left (right) ideal of S.

Theorem 3.14:

If $\mathcal{A} = < \bar{\mu}, \gamma >$ is a cubic left (right) ideal of S then $\mathcal{A}^c = < (\bar{\mu})^c, (\gamma)^c >$ is also a cubic left (right) ideal of S.

Proof:

Let x, y $\in$ S and $\mathcal{A} = < \bar{\mu}, \gamma >$ is a cubic left (right) ideal of S, then

$i)$ $(\bar{\mu})^c(x-y) = 1 - \bar{\mu}(x-y) \leq 1 - \min\{\bar{\mu}(x), \bar{\mu}(y)\} = \max\{1 - \bar{\mu}(x), 1 - \bar{\mu}(y)\}$
$(\bar{\mu})^c(x-y) \leq \max\{(\bar{\mu})^c(x), (\bar{\mu})^c(y)\}$

$ii)$ $(\gamma)^c(x-y) = 1 - \gamma(x-y) \geq 1 - \max\{\gamma(x), \gamma(y)\} = \min\{1 - \gamma(x), 1 - \gamma(y)\}$
$(\gamma)^c(x-y) \geq \min\{(\gamma^c(x), (\gamma^c(y)\}$

$iii)$ $(\bar{\mu})^c(xy) = 1 - \bar{\mu}(xy) \leq 1 - \min\{\bar{\mu}(x), \bar{\mu}(y)\} = \max\{1 - \bar{\mu}(x), 1 - \bar{\mu}(y)\}$
$(\bar{\mu})^c(xy) \leq \max\{(\bar{\mu})^c(x), (\bar{\mu})^c(y)\}$

$iv)$ $(\gamma)^c(xy) = 1 - \gamma(xy) \geq 1 - \max\{\gamma(x), \gamma(y)\} = \min\{1 - \gamma(x), 1 - \gamma(y)\}$
$(\gamma)^c(xy) \geq \min\{(\gamma^c(x), (\gamma^c(y)\}$

$v)$ $(\bar{\mu})^c(xy) = 1 - \bar{\mu}(xy) \leq 1 - \bar{\mu}(y) = (\bar{\mu})^c(y)$

$iv)$ $(\gamma)^c(xy) = 1 - \gamma(xy) \geq 1 - \gamma(y) \geq (\gamma)^c(y)$

Hence $\mathcal{A}^c = < (\bar{\mu})^c, (\gamma)^c >$ is a cubic left (right) ideal of S.

4. Homomorphism of Cubic Ideals in Subtraction Semigroups:

Definition 4.1 [2] Let P and Q be two subtraction semigroups of S and f: P → Q is called a subtraction semigroup homomorphism. If
i) f(x – y) = f(x) – f(y)
ii) f(x y) = f(x) f(y) $\forall\, x, y \in P$.

Definition 4.2 [7] Let P and Q be given classical sets. A mapping f: P→Q induces two mappings C_f: C(P)→ $C(Q)$, $\mathcal{A}_1 \to C_f(\mathcal{A}_1)$ and C_f^{-1}: $C(Q) \to C(P)$, $\mathcal{A}_2 \to C_f^{-1}(\mathcal{A}_2)$. Where the mapping C_f is called cubic transformation and C_f^{-1} is called inverse cubic transformation.

Definition 4.3 [7] A cubic set $\mathcal{A} = < \bar{\mu}, \gamma >$ in P has the cubic property if for any subset X of P there exist $x_0 \in X$ such that $\bar{\mu}(x_0) = \sup_{x \in X} \bar{\mu}(x)$ and $\gamma(x_0) = \inf_{x \in X} \gamma(x)$

Definition 4.4 Let f be a mapping from a set P to a set Q and $\mathcal{A} =< \bar{\mu}, \gamma >$ be a cubic set of P then the image of P $C_f(\mathcal{A}) = < C_f(\bar{\mu}), C_f(\gamma) >$ is a cubic set of Q is defined by

$$C_f(\mathcal{A})(x) = \begin{cases} C_f(\bar{\mu})(x) = \begin{cases} \sup_{y\in f^{-1}(x)} \bar{\mu}(y) & if\ f^{-1}(x) \neq \emptyset \\ [0,0] & otherwise \end{cases} \\ C_f(\gamma)(x) = \begin{cases} \inf_{y\in f^{-1}(x)} \gamma(y) & if\ f^{-1}(x) \neq \emptyset \\ 1 & otherwise \end{cases} \end{cases}$$

Let f be a mapping from a set P to Q and $\mathcal{A} =< \bar{\mu}, \gamma >$ be a cubic set of Q then the pre image of Q $C_f^{-1}(\mathcal{A}) = < C_f^{-1}(\bar{\mu}), C_f^{-1}(\gamma) >$ is a cubic set of P is defined by

$$C_f^{-1}(\mathcal{A}) = \begin{cases} C_f^{-1}(\bar{\mu}(x)) = \bar{\mu}(f(x)) \\ C_f^{-1}(\gamma(x)) = \gamma(f(x)) \end{cases}$$

Theorem 4.5:

Let P and Q be two subtraction semigroups, let f : P → Q be an onto homomorphism of subtraction semigroups and C_f : C(P)→ C(Q) be the cubic transformation induced by f. If $\mathcal{A} = < \bar{\mu}, \gamma >$ is a cubic left (right) ideal of P by the cubic property then $C_f(\mathcal{A}) = < C_f(\bar{\mu}), C_f(\gamma) >$ is a cubic left(right) ideal of Q.

Proof:

Given $f(x), f(y), f(x-y) \in f(P)$

let $x_0 \in f^{-1}(f(x)), y_0 \in f^{-1}(f(y))$ *and* $x_0 - y_0 \in f^{-1}(f(x) - f(y))$ be such that

$\bar{\mu}(x_0 - y_0) = \sup_{a\in f^{-1}(f(x)-f(y))} \bar{\mu}(a), \quad \gamma(x_0 - y_0) = \inf_{a\in f^{-1}(f(x)-f(y))} \gamma(a)$

$\bar{\mu}(x_0) = \sup_{b\in f^{-1}(f(x))} \bar{\mu}(b),\ \gamma(x_0) = \inf_{b\in f^{-1}(f(x))} \gamma(b)$ and

$\bar{\mu}(y_0) = \sup_{c\in f^{-1}(f(y))} \bar{\mu}(c),\ \gamma(y_0) = \inf_{c\in f^{-1}(f(y))} \gamma(c)$ then

$$\begin{aligned} C_f(\bar{\mu})(f(x)) &= \sup_{z\in f^{-1}(f(x))} \bar{\mu}(z) \\ &\geq \bar{\mu}(x_0) \\ &\geq \min\{\bar{\mu}(x_0 - y_0), \bar{\mu}(y_0)\} \\ &= \min\{\sup_{a\in f^{-1}(f(x)-f(y))} \bar{\mu}(a), \sup_{c\in f^{-1}(f(y))} \bar{\mu}(c)\} \\ &= \min\{C_f(\bar{\mu})(f(x) - f(y)), C_f(\bar{\mu})(f(y))\} \\ &= \min\{C_f(\bar{\mu})(f(x-y)), C_f(\bar{\mu})(f(y))\} \end{aligned}$$

$$\begin{aligned} C_f(\gamma)(f(x)) &= \inf_{z\in f^{-1}(f(x))} \gamma(z) \\ &\leq \gamma(x_0) \\ &\leq \max\{\gamma(x_0 - y_0), \gamma(y_0)\} \\ &= \max\{\inf_{a\in f^{-1}(f(x)-f(y))} \gamma(a), \inf_{c\in f^{-1}(f(y))} \gamma(c)\} \\ &= \max\{C_f(\gamma)(f(x) - f(y)), C_f(\gamma)(f(y))\} \\ &= \max\{C_f(\gamma)(f(x-y)), C_f(\gamma)(f(y))\} \end{aligned}$$

$$\begin{aligned} C_f(\bar{\mu})(f(x)f(y)) &= \sup_{z\in f^{-1}(f(x)f(y))} \bar{\mu}(z) \\ &\geq \bar{\mu}(x_0 y_0) \\ &\geq \min\{\bar{\mu}(x_0), \bar{\mu}(y_0)\} \\ &= \min\{\sup_{b\in f^{-1}(f(x))} \bar{\mu}(a), \sup_{c\in f^{-1}(f(y))} \bar{\mu}(c)\} \\ &= \min\{C_f(\bar{\mu})(f(x)), C_f(\bar{\mu})(f(y))\} \end{aligned}$$

$$\begin{aligned} C_f(\gamma)(f(x)f(y)) &= \inf_{z\in f^{-1}(f(x)f(y))} \gamma(z) \\ &\leq \gamma(x_0 y_0) \\ &\leq \max\{\gamma(x_0), \gamma(y_0)\} \\ &= \max\{\inf_{b\in f^{-1}(f(x))} \gamma(b), \inf_{c\in f^{-1}(f(y))} \gamma(c)\} \\ &= \max\{C_f(\gamma)(f(x)), C_f(\gamma)(f(y))\} \end{aligned}$$

$$\begin{aligned} C_f(\bar{\mu})(f(x)f(y)) &= \sup_{z\in f^{-1}(f(x)f(y))} \bar{\mu}(z) \\ &\geq \bar{\mu}(x_0 y_0) \\ &\geq \bar{\mu}(y_0) \\ &\geq \sup_{c\in f^{-1}(f(y))} \bar{\mu}(c) \\ &= C_f(\bar{\mu})(f(y)) \end{aligned}$$

$$\begin{aligned} C_f(\gamma)(f(x)f(y)) &= \inf_{z\in f^{-1}(f(x)f(y))} \gamma(z) \\ &\leq \gamma(x_0 y_0) \\ &\leq \gamma(y_0) \\ &\leq \inf_{c\in f^{-1}(f(y))} \gamma(c) \end{aligned}$$

$= C_f(\gamma)(f(y))$

Hence $C_f(\mathcal{A}) = < C_f(\bar{\mu}), C_f(\gamma) >$ is a cubic left (right) ideal of Q.

Theorem 4.6:

Let P and Q be two subtraction semigroups, let f: P→Q be an onto homomorphism of subtraction semigroups and $C_f^{-1}: C(Q) \to C(P)$ be the inverse cubic transformation induced by f.

If $\mathcal{A} = < \bar{\mu}, \gamma >$ is a cubic left (right) ideal of Q by the cubic property then $C_f^{-1}(\mathcal{A}) = < C_f^{-1}(\bar{\mu}), C_f^{-1}(\gamma) >$ is a cubic left (right) ideal of P.

Proof:

Let $\mathcal{A} = < \bar{\mu}, \gamma >$ is a cubic left (right) ideal of Q and let x, y ∈ P then

$i)$ $C_f^{-1}(\bar{\mu}(x)) = \bar{\mu}(f(x)) \geq \min\{\bar{\mu}(f(x) - f(y)), \bar{\mu}(f(y))\} = \min\{\bar{\mu}(f(x - y)), \bar{\mu}(f(y))\}$

$C_f^{-1}(\bar{\mu}(x)) \geq \min\{C_f^{-1}(\bar{\mu}(x - y)), C_f^{-1}(\bar{\mu}(y))\}$

$ii)$ $C_f^{-1}(\gamma(x)) = \gamma(f(x)) \leq \max\{\gamma(f(x) - f(y)), \gamma(f(y))\} = \max\{\gamma(f(x - y)), \gamma(f(y))\}$

$C_f^{-1}(\gamma(x)) \leq \max\{C_f^{-1}(\gamma(x - y)), C_f^{-1}(\gamma(y))\}$

$iii)$ $C_f^{-1}(\bar{\mu}(xy)) = \bar{\mu}(f(xy) = \bar{\mu}(f(x)f(y)) \geq \min\{\bar{\mu}(f(x)), \bar{\mu}(f(y))\}$

$C_f^{-1}(\bar{\mu}(xy)) \geq \min\{C_f^{-1}(\bar{\mu}(x)), C_f^{-1}(\bar{\mu}(y))\}$

$iv)$ $C_f^{-1}(\gamma(xy)) = \gamma(f(xy) = \gamma(f(x)f(y)) \leq \max\{\gamma(f(x)), \gamma(f(y))\}$

$C_f^{-1}(\gamma(xy)) \leq \max\{C_f^{-1}(\gamma(x)), C_f^{-1}(\gamma(y))\}$

$v)$ $C_f^{-1}(\bar{\mu}(xy)) = \bar{\mu}(f(xy) = \bar{\mu}(f(x)f(y)) \geq \bar{\mu}(f(y)) \geq C_f^{-1}(\bar{\mu}(y))$

$vi)$ $C_f^{-1}(\gamma(xy)) = \gamma(f(xy) = \gamma(f(x)f(y)) \leq \gamma(f(y)) \leq C_f^{-1}((y))$

Hence $C_f^{-1}(\mathcal{A}) = < C_f^{-1}(\bar{\mu}), C_f^{-1}(\gamma) >$ is a cubic left (right) ideals of P.

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